

Mixed hook-length formula for degenerate affine Hecke algebras

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In this article we work with the degenerate affine Hecke algebra H_l corresponding to the general linear group GL_l over a local non-Archimedean field. This algebra was introduced by V. Drinfeld in [D], see also [L]. The complex associative algebra H_l is generated by the symmetric group algebra $\mathbb{C}S_l$ and by the pairwise commuting elements x_1, \dots, x_l with the cross relations for $p = 1, \dots, l-1$ and $q = 1, \dots, l$

$$\begin{aligned}\sigma_{p,p+1} x_q &= x_q \sigma_{p,p+1}, \quad q \neq p, p+1; \\ \sigma_{p,p+1} x_p &= x_{p+1} \sigma_{p,p+1} - 1.\end{aligned}$$

Here and in what follows $\sigma_{pq} \in S_l$ denotes transposition of the numbers p and q .

For any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of l let V_λ be the corresponding irreducible $\mathbb{C}S_l$ -module. There is a homomorphism $H_l \rightarrow \mathbb{C}S_l$ identical on the subalgebra $\mathbb{C}S_l \subset H_l$ such that $x_p \mapsto \sigma_{1p} + \dots + \sigma_{p-1,p}$ for each $p = 1, \dots, l-1$. So V_λ can be regarded as a H_l -module. For any number $z \in \mathbb{C}$ there is also an automorphism of H_l identical on the subalgebra $\mathbb{C}S_l \subset H_l$ such that $x_p \mapsto x_p + z$ for each $p = 1, \dots, l$. We will denote by $V_\lambda(z)$ the H_l -module obtained by pulling V_λ back through this automorphism. The module $V_\lambda(z)$ is irreducible by definition.

Consider the algebra $H_l \otimes H_m$ where both l and m are positive integers. It is isomorphic to the subalgebra in H_{l+m} generated by the transpositions σ_{pq} where $1 \leq p < q \leq l$ or $l+1 \leq p < q \leq l+m$, along with all the elements x_1, \dots, x_{l+m} . For any partition μ of m and any number $w \in \mathbb{C}$ take the corresponding H_m -module $V_\mu(w)$. Now consider the H_{l+m} -module W induced from the $H_l \otimes H_m$ -module $V_\lambda(z) \otimes V_\mu(w)$. Also consider the H_{l+m} -module W' induced from the $H_m \otimes H_l$ -module $V_\mu(w) \otimes V_\lambda(z)$. Suppose that $z - w \notin \mathbb{Z}$, then the modules W and W' are irreducible and equivalent; see [C]. So there is a unique, up to scalar multiplier, H_{l+m} -intertwining operator $I : W \rightarrow W'$. For a certain particular realization of the modules W and W' we will give an explicit expression for the operator I . In particular, this will fix the normalization of I .

Following [C], we will realize W and W' as certain left ideals in the group algebra $\mathbb{C}S_{l+m}$. We will have $W' = W\tau$ where $\tau \in S_{l+m}$ is the permutation

$$(1, \dots, m, m+1, \dots, m+l) \mapsto (l+1, \dots, l+m, 1, \dots, l).$$

Let $J : W \rightarrow W$ be the composition of the operator I and the operator $W' \rightarrow W$ of the right multiplication by τ^{-1} . In §2 we give an explicit expression for the operator I . Using this expression makes the spectral analysis of the operator J an arduous task; cf. [AK]. However, our results are based on this expression.

The subalgebra $\mathbb{C}S_{l+m} \subset H_{l+m}$ acts in $W \subset \mathbb{C}S_{l+m}$ via left multiplication. Under this action the space W splits into irreducible components according to the Littlewood-Richardson rule [M]. The operator J commutes with this action. Hence J acts via multiplication by a certain complex number in every irreducible component of W appearing with multiplicity one. In this article we compute these numbers for certain multiplicity free components of W ; see Theorems 1 and 2.

For example, there are two distinguished irreducible components of the $\mathbb{C}S_{l+m}$ -module W which always have multiplicity one. They correspond to the partitions

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots) \quad \text{and} \quad (\lambda' + \mu')' = (\lambda'_1 + \mu'_1, \lambda'_2 + \mu'_2, \dots)'$$

where as usual $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ denotes the conjugate partition. Denote by $h_{\lambda\mu}(z, w)$ the ratio of the corresponding two eigenvalues of J , this ratio does not depend on the normalization of the operator I . Theorems 1 and 2 have the following

Corollary:

$$h_{\lambda\mu}(z, w) = \prod_{i,j} \frac{z - w - \lambda'_j - \mu_i + i + j - 1}{z - w + \lambda_i + \mu'_j - i - j + 1}$$

where the product is taken over all $i, j = 1, 2, \dots$ such that $j \leq \lambda_i, \mu_i$.

We derive this corollary in §6. Now identify partitions with their Young diagrams. The condition $j \leq \lambda_i, \mu_i$ means that the box (i, j) belongs to the intersection of the diagrams λ and μ . If $\lambda = \mu$ the numbers $\lambda_i + \lambda'_j - i - j + 1$ are called [M] the *hook-lengths* of the diagram λ . If $\lambda \neq \mu$ the numbers in the above fraction

$$\lambda_i + \mu'_j - i - j + 1 \quad \text{and} \quad \lambda'_j + \mu_i - i - j + 1$$

may be called the *mixed hook-lengths* of the first and of the second kind respectively. Both these numbers are positive for any box (i, j) in the intersection of λ and μ .

Let h_λ denote the product of all l hook-lengths of the Young diagram λ . This product appears in the well-known *hook-length formula* [M] for the dimension of the irreducible $\mathbb{C}S_l$ -module V_λ : $\dim V_\lambda = l! / h_\lambda$. In §7 we derive this formula from our Theorem 2. At the end of §7 we discuss applications of our Theorems 1 and 2 to the representation theory of affine Hecke algebras, cf. [LNT].

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§1. Here we will collect several known facts about the irreducible $\mathbb{C}S_l$ -modules. Fix the chain $S_1 \subset S_2 \subset \dots \subset S_l$ of subgroups with the standard embeddings. There is a distinguished basis in the space V_λ associated with this chain, called the *Young basis*. Its vectors are labelled by the standard tableaux [M] of shape λ . For every such a tableau Λ the basis vector $v_\Lambda \in V_\lambda$ is defined, up to a scalar multiplier, as follows. For any $p = 1, \dots, l-1$ take the tableau obtained from Λ by removing each of the numbers $p+1, \dots, l$. Let the Young diagram π be its shape. Then the vector v_Λ is contained in an irreducible $\mathbb{C}S_p$ -submodule of V_λ corresponding to π . Fix an S_l -invariant inner product $\langle \cdot, \cdot \rangle$ in V_λ . The vectors v_Λ are then pairwise orthogonal. We will agree that $\langle v_\Lambda, v_\Lambda \rangle = 1$ for every Λ .

There is an alternative definition [J] of the vector $v_\Lambda \in V_\lambda$. For each $p = 1, \dots, l$ put $c_p = j - i$ if the number p appears in the i -row and the j -th column of the tableau Λ . The number c_p is called the *content* of the box of the diagram λ occupied by p . Here on the left we show the column tableau of shape $\lambda = (3, 2)$:

1	3	5
2	4	

0	1	2
-1	0	

On the right we indicated the contents of the boxes of the Young diagram $\lambda = (3, 2)$. So here we have $(c_1, \dots, c_5) = (0, -1, 1, 0, 2)$. For each $p = 1, \dots, l$ consider the image $\sigma_{1p} + \dots + \sigma_{p-1,p}$ of the generator x_p under the homomorphism $H_l \rightarrow \mathbb{C}S_l$.

Proposition 1 ([J]): *the element v_Λ of the $\mathbb{C}S_l$ -module V_λ is determined, up to a scalar multiplier, by equations $(\sigma_{1p} + \dots + \sigma_{p-1,p}) \cdot v_\Lambda = c_p v_\Lambda$ for $p = 1, \dots, l$.*

Take the diagonal matrix element of V_λ corresponding to the vector v_Λ

$$F_\Lambda = \sum_{\sigma \in S_l} \langle v_\Lambda, \sigma \cdot v_\Lambda \rangle \sigma.$$

As a general property of matrix elements, we have the equality $F_\Lambda^2 = l! / \dim V_\lambda \cdot F_\Lambda$. There is an alternative expression for the element $F_\Lambda \in \mathbb{C}S_l$, it goes back to [C]. For any distinct $p, q = 1, \dots, l$ introduce the rational function of two complex variables u, v

$$f_{pq}(u, v) = 1 - \sigma_{pq} / (u - v).$$

These functions take values in the algebra $\mathbb{C}S_l$ and satisfy the relations

$$f_{pq}(u, v) f_{pr}(u, w) f_{qr}(v, w) = f_{qr}(v, w) f_{pr}(u, w) f_{pq}(u, v) \quad (1)$$

$$f_{pq}(u, v) f_{qp}(v, u) = 1 - (u - v)^{-2}. \quad (2)$$

for all pairwise distinct indices p, q, r . Introduce l complex variables z_1, \dots, z_l . Order lexicographically the pairs (p, q) with $1 \leq p < q \leq l$. Define the rational function $F_\Lambda(z_1, \dots, z_l)$ as the ordered product of the functions $f_{pq}(z_p + c_p, z_q + c_q)$ over all the pairs (p, q) . Let \mathcal{Z}_Λ be the vector subspace in \mathbb{C}^l consisting of all l -tuples (z_1, \dots, z_l) such that $z_p = z_q$, whenever the numbers p and q appear in the same row of the tableau Λ .

Proposition 2 ([N]): *the restriction of the rational function $F_\Lambda(z_1, \dots, z_l)$ to \mathcal{Z}_Λ is regular at the origin $(0, \dots, 0) \in \mathbb{C}^l$, and takes there the value F_Λ .*

§2. Let us choose any standard tableau Λ of shape λ . The H_l -module $V_\lambda(z)$ can be realized as the left ideal in $\mathbb{C}S_l$ generated by the element F_Λ . The subalgebra $\mathbb{C}S_l \subset H_l$ acts here via left multiplication. Due to Proposition 1 and to the defining relations of H_l , the action of the generators x_1, \dots, x_l in this left ideal can be then determined by setting $x_p \cdot F_\Lambda = (c_p + z) F_\Lambda$ for each $p = 1, \dots, l$.

Also fix any standard tableau M of shape μ . Let d_q be the content of the box of the diagram μ occupied by $q = 1, \dots, m$ in M . We will denote by \overline{G} the image of any element $G \in \mathbb{C}S_m$ under the embedding $\mathbb{C}S_m \rightarrow \mathbb{C}S_{l+m} : \sigma_{pq} \mapsto \sigma_{l+p, l+q}$. The H_{l+m} -module W induced from the $H_l \otimes H_m$ -module $V_\lambda(z) \otimes V_\mu(w)$ can be realized as the left ideal in $\mathbb{C}S_{l+m}$ generated by the product $F_\Lambda \overline{F}_M$. The action of the generators x_1, \dots, x_{l+m} in the latter left ideal can be determined by setting

$$x_p \cdot F_\Lambda \overline{F}_M = (c_p + z) F_\Lambda \overline{F}_M \quad \text{for each } p = 1, \dots, l; \quad (3)$$

$$x_{l+q} \cdot F_\Lambda \overline{F}_M = (d_q + w) F_\Lambda \overline{F}_M \quad \text{for each } q = 1, \dots, m.$$

Now introduce the ordered products in the symmetric group algebra $\mathbb{C}S_{l+m}$

$$R_{\Lambda M}(z, w) = \overrightarrow{\prod}_{p=1, \dots, l} \left(\overleftarrow{\prod}_{q=1, \dots, m} f_{p, l+q}(c_p + z, d_q + w) \right),$$

$$R'_{\Lambda M}(z, w) = \overleftarrow{\prod} \left(\overrightarrow{\prod} f_{p, l+q}(c_p + z, d_q + w) \right);$$

the arrows indicate ordering of (non-commuting) factors. We keep to the assumption $z - w \notin \mathbb{Z}$. Applying Proposition 2 to the tableaux Λ, M and using (1) repeatedly, we get

$$F_\Lambda \bar{F}_M R_{\Lambda M}(z, w) = R'_{\Lambda M}(z, w) F_\Lambda \bar{F}_M. \quad (4)$$

Hence the right multiplication in $\mathbb{C}S_{l+m}$ by $R_{\Lambda M}(z, w)$ preserves the left ideal W .

The H_{l+m} -module W' induced from the $H_m \otimes H_l$ -module $V_\mu(w) \otimes V_\lambda(z)$ can be then realized as the left ideal in $\mathbb{C}S_{l+m}$ generated by the element $\tau^{-1} F_\Lambda \bar{F}_M \tau$, so that $W' = W\tau$. The action of the generators x_1, \dots, x_{l+m} in W' is determined by

$$x_q \cdot \tau^{-1} F_\Lambda \bar{F}_M \tau = (d_q + w) \tau^{-1} F_\Lambda \bar{F}_M \tau \quad \text{for each } q = 1, \dots, m; \quad (5)$$

$$x_{m+p} \cdot \tau^{-1} F_\Lambda \bar{F}_M \tau = (c_p + z) \tau^{-1} F_\Lambda \bar{F}_M \tau \quad \text{for each } p = 1, \dots, l. \quad (6)$$

Consider the operator of right multiplication in $\mathbb{C}S_{l+m}$ by $R_{\Lambda M}(z, w)\tau$. Denote by I the restriction of this operator to the subspace $W \subset \mathbb{C}S_{l+m}$. Due to (4) the image of the operator I is contained in the subspace W' .

Proposition 3: *the operator $I : W \rightarrow W'$ commutes with the action of H_{l+m} .*

Proof. The subalgebra $\mathbb{C}S_{l+m} \subset H_{l+m}$ acts in W, W' via left multiplication; the operator I commutes with this action by definition. The left ideal W is generated by the element $F_\Lambda \bar{F}_M$, so it suffices to check that $x_p \cdot I(F_\Lambda \bar{F}_M) = I(x_p \cdot F_\Lambda \bar{F}_M)$ for each $p = 1, \dots, l + m$. Firstly consider the case $1 \leq p \leq l$, then by (3,4,5,6)

$$\begin{aligned} x_p \cdot I(F_\Lambda \bar{F}_M) &= x_p \cdot (R'_{\Lambda M}(z, w)\tau)(\tau^{-1} F_\Lambda \bar{F}_M \tau) \\ &= (R'_{\Lambda M}(z, w)\tau)(x_{\tau^{-1}(p)} \cdot \tau^{-1} F_\Lambda \bar{F}_M \tau) \\ &= (R'_{\Lambda M}(z, w)\tau)(c_p + z)(\tau^{-1} F_\Lambda \bar{F}_M \tau) = I(x_p \cdot F_\Lambda \bar{F}_M); \end{aligned}$$

here we also used the defining relations of the algebra H_{l+m} . For more details of this argument see [L]. The case $l + 1 \leq p \leq l + m$ can be considered similarly \square

Consider the operator of the right multiplication in $\mathbb{C}S_{l+m}$ by $R_{\Lambda M}(z, w)$. This operator preserves the subspace W due to (4). The restriction of this operator to W will be denoted by J . The subalgebra $\mathbb{C}S_{l+m} \subset H_{l+m}$ acts in W via left multiplication, and J commutes with this action. Regard W as a $\mathbb{C}S_{l+m}$ -module only. Let ν be any partition of $l + m$ such that the irreducible $\mathbb{C}S_{l+m}$ -module V_ν appears in W with multiplicity one. The operator $J : W \rightarrow W$ preserves the subspace $V_\nu \subset W$ and acts there as multiplication by a certain number from \mathbb{C} . Denote this number by $r_\nu(z, w)$; it depends on z and w as a rational function of $z - w$, and does not depend on the choice of the tableaux Λ and M . Our aim is to compute the eigenvalues $r_\nu(z, w)$ of J for certain ν .

Before performing the computation, let us observe one general property of the eigenvalues $r_\nu(z, w)$. Similarly to the definition of the H_{l+m} -intertwining operator $I : W \rightarrow W'$, one can define an operator $I' : W' \rightarrow W$ as the restriction to W' of the operator of the right multiplication in $\mathbb{C}S_{l+m}$ by $R_{M\Lambda}(w, z)\tau^{-1}$. The operator I' commutes with the action of H_{l+m} as well. One can also consider the operator $J' : W' \rightarrow W'$, defined as the restriction to W' of the operator of right multiplication in $\mathbb{C}S_{l+m}$ by $R_{M\Lambda}(w, z)$. There is a unique irreducible $\mathbb{C}S_{l+m}$ -submodule in $V'_\nu \subset W'$ equivalent to $V_\nu \subset W$; actually here we have $V'_\nu = V_\nu \tau$. Consider the corresponding eigenvalue $r'_\nu(w, z)$ of the operator J' .

Proposition 4:

$$r_\nu(z, w) r'_\nu(w, z) = \prod_{i=1}^{\lambda'_1} \prod_{k=1}^{\mu'_1} \frac{(z - w + \lambda_i - i + k)(z - w - \mu_k - i + k)}{(z - w + \lambda_i - \mu_k - i + k)(z - w - i + k)}.$$

Proof. The product $r_\nu(z, w) r'_\nu(w, z)$ is the eigenvalue of the composition $I' \circ I : W \rightarrow W$, corresponding to the subspace $V_\nu \subset W$. By definition, this composition is the operator of right multiplication in $W \subset \mathbb{C}S_{l+m}$ by the element

$$\begin{aligned} R_{\Lambda M}(z, w) \tau R_{M\Lambda}(w, z) \tau^{-1} &= \prod_{p=1, \dots, l}^{\rightarrow} \left(\prod_{q=1, \dots, m}^{\leftarrow} f_{p, l+q}(c_p + z, d_q + w) \right) \cdot \tau \times \\ &\quad \prod_{q=1, \dots, m}^{\rightarrow} \left(\prod_{p=1, \dots, l}^{\leftarrow} f_{q, m+p}(d_q + w, c_p + z) \right) \cdot \tau^{-1} = \\ &\quad \prod_{p=1, \dots, l}^{\rightarrow} \left(\prod_{q=1, \dots, m}^{\leftarrow} f_{p, l+q}(c_p + z, d_q + w) \right) \cdot \prod_{p=1, \dots, l}^{\leftarrow} \left(\prod_{q=1, \dots, m}^{\rightarrow} f_{l+q, p}(d_q + w, c_p + z) \right) \\ &= \prod_{p=1}^l \prod_{q=1}^m \left(1 - (z - w + c_p - d_q)^{-2} \right). \end{aligned}$$

The last equality has been obtained by using repeatedly the relations (2), it shows that the composition $I' \circ I$ is a scalar operator. Now recall that the contents of the boxes in the same row of a Young diagram increase by 1, when moving from left to right. For the i -th row of λ , the contents of the leftmost and rightmost boxes are $1 - i$ and $\lambda_i - i$ respectively. For the k -th row of μ , the contents of the leftmost and rightmost boxes are respectively $1 - k$ and $\mu_k - k$. Hence the right hand side of the last equality can be rewritten as

$$\begin{aligned} &\prod_{p=1}^l \prod_{q=1}^m \left(\frac{z - w + c_p - d_q + 1}{z - w + c_p - d_q} \cdot \frac{z - w + c_p - d_q - 1}{z - w + c_p - d_q} \right) = \\ &\prod_{i=1}^{\lambda'_1} \prod_{q=1}^m \left(\frac{z - w + \lambda_i - i - d_q + 1}{z - w + 1 - i - d_q} \cdot \frac{z - w - i - d_q}{z - w + \lambda_i - i - d_q} \right) = \\ &\prod_{i=1}^{\lambda'_1} \prod_{k=1}^{\mu'_1} \left(\frac{z - w + \lambda_i - i + k}{z - w + \lambda_i - i - \mu_k + k} \cdot \frac{z - w - i - \mu_k + k}{z - w - i + k} \right) \quad \square \end{aligned}$$

§3. Choose any sequence $a_1, \dots, a_{\lambda'_1} \in \{1, 2, \dots\}$ of pairwise distinct indices, we emphasize that this sequence needs not to be increasing. Here λ'_1 is the number of non-zero parts in the partition λ . Consider the partition μ as an infinite sequence with finitely many non-zero terms. Define an infinite sequence $\gamma = (\gamma_1, \gamma_2, \dots)$ by

$$\gamma_{a_i} = \mu_{a_i} + \lambda_i, \quad i = 1, \dots, \lambda'_1;$$

Suppose we have $\gamma_1 \geq \gamma_2 \geq \dots$, so that γ is a partition of $l + m$. Then the irreducible $\mathbb{C}S_{l+m}$ -module V_γ corresponding to the partition γ appears in W with multiplicity one. Indeed, the multiplicity of V_γ in W equals the multiplicity of V_λ in the $\mathbb{C}S_l$ -module corresponding to the skew Young diagram γ/μ . The latter multiplicity is one by the definition of γ ; see for instance [M].

We will evaluate the number $r_\gamma(z, w)$ by applying the operator J to a particular vector in the subspace $V_\gamma \subset W$. Assume that $\Lambda = \Lambda^c$ is the column tableau of shape λ ; the tableau M will be still arbitrary. The image of the action of the element $F_{\Lambda^c} \bar{F}_M$ in the irreducible $\mathbb{C}S_{l+m}$ -module V_γ is a one-dimensional subspace. Let us describe this subspace explicitly. The standard chain of subgroups $S_1 \subset S_2 \subset \dots \subset S_l$ corresponds to the natural ordering of the numbers $1, 2, \dots, l$. Now consider the new chain of subgroups

$$S_1 \subset \dots \subset S_m \subset S_{1+m} \subset \dots \subset S_{l+m}$$

corresponding to the ordering $l+1, \dots, l+m, 1, \dots, l$. Notice that the element \bar{F}_M belongs to the subgroup S_m in this new chain. Take the Young basis in the space V_γ associated with the new chain. In particular, take the basis vector $v_\Gamma \in V_\gamma$ corresponding to the tableau Γ of shape γ defined as follows. The numbers $l+1, \dots, l+m$ appear in Γ respectively in the same positions as the numbers $1, \dots, m$ do in tableau M . Now for every positive integer j consider all those parts of λ which are equal to j . These are λ_i with $i = \lambda'_{j+1} + 1, \lambda'_{j+1} + 2, \dots, \lambda'_j$. Let $f_1, \dots, f_{\lambda'_j - \lambda'_{j+1}}$ be the indices a_i with $\lambda_i = j$, arranged in the increasing order. By definition, the numbers appearing in the rows $\lambda'_{j+1} + 1, \lambda'_{j+1} + 2, \dots, \lambda'_j$ of the tableau Λ^c , will stand in the rows $f_1, \dots, f_{\lambda'_j - \lambda'_{j+1}}$ of Γ respectively. For example, here for $\lambda = (3, 2)$ and $\mu = (2, 1)$ with $a_1 = 2$ and $a_2 = 1$ we show a standard tableau M and the corresponding tableau Γ :

1	2
3	

6	7	2	4
8	1	3	5

Proposition 5: *with respect to the ordering $l+1, \dots, l+m, 1, \dots, l$ the tableau Γ is standard.*

Proof. Reading the rows of the tableau Γ from left to right, or reading its columns downwards, the numbers $l+1, \dots, l+m$ appear in the increasing order because M is standard. These numbers will also appear before $1, \dots, l$. Moreover, the numbers $1, \dots, l$ increase along each row of the tableau Γ by the definition of Λ^c . Now suppose that a column of Γ contains two different numbers $p, q \in \{1, \dots, l\}$. Let a, \bar{a} be the corresponding rows; assume that $a < \bar{a}$. Then $a = a_i$ and $\bar{a} = a_{\bar{i}}$ for certain indices $i, \bar{i} \in \{1, \dots, \lambda'_1\}$. If $\lambda_i \geq \lambda_{\bar{i}}$ then $p < q$ by definition of Λ^c .

Let j, \bar{j} be the columns corresponding to the numbers p, q in the tableau Λ^c . Suppose that $\lambda_i < \lambda_{\bar{i}}$, then $\mu_a > \mu_{\bar{a}}$ because $\mu_a + \lambda_i \geq \mu_{\bar{a}} + \lambda_{\bar{i}}$. Since p and q stand in the same column of the tableau Γ , we then have $j < \bar{j}$ and $p < q$ \square

Proposition 6: *the one-dimensional subspace $F_{\Lambda^c} \bar{F}_M \cdot V_\gamma \subset V_\gamma$ is obtained from the space $\mathbb{C}v_\Gamma$ by antisymmetrization relative to the columns of the tableau Λ^c .*

Proof. Let S_λ be the subgroup in S_l consisting of all permutations which preserve the columns of the tableau Λ^c as sets. Let $Q \in \mathbb{C}S_l$ be the alternated sum of all

elements from S_λ . Put $V = \bar{F}_M \cdot V_\gamma$. The subspace $V \subset V_\gamma$ is spanned by the Young vectors, corresponding to the tableaux which agree with Γ in the entries $l+1, \dots, l+m$. The action of the element F_{Λ^c} in V_γ preserves the subspace V , and the image $F_{\Lambda^c} \cdot V \subset V$ is one-dimensional. Moreover, we have $F_{\Lambda^c} \cdot V = Q \cdot V$; see [JK]. It now remains to check that $Q \cdot v_\Gamma \neq 0$.

By our choice of the tableau Γ , it suffices to consider the case when λ consists of one column only. But then the element $Q \in \mathbb{C}S_l$ is central. On the other hand, the vector $v_\Gamma \in V$ is $\mathbb{C}S_l$ -cyclic; see [C]. So $Q \cdot V \neq \{0\}$ implies $Q \cdot v_\Gamma \neq 0$ \square

§4. Let γ be any of the partitions of $l+m$ described in the beginning of §3. Our main result is the following expression for the corresponding eigenvalue $r_\gamma(z, w)$ of the operator $J : W \rightarrow W$. This expression will be obtained by applying J to the vector $F_{\Lambda^c} \bar{F}_M \cdot v_\Gamma$ in $V_\gamma \subset W$ and using Proposition 6.

Theorem 1:

$$r_\gamma(z, w) = \prod_{(i,j)} \frac{z - w - \lambda'_j - \mu_{a_i} + a_i + j - 1}{z - w - i + j}$$

where the product is taken over all boxes (i, j) of the Young diagram λ .

Proof. Using (4) and applying Proposition 2.12 of [N] to the tableau M , we obtain the equalities in the algebra $\mathbb{C}S_{l+m}$

$$\begin{aligned} F_{\Lambda^c} \bar{F}_M R_{\Lambda^c M}(z, w) &= \dim V_\lambda / l! \cdot F_{\Lambda^c} \bar{F}_M R_{\Lambda^c M}(z, w) F_{\Lambda^c} = \dim V_\lambda / l! \times \\ &F_{\Lambda^c} \bar{F}_M \left(\prod_{p=1, \dots, l}^{\rightarrow} \frac{\sigma_{l+1,p} + \dots + \sigma_{l+m,p} - c_p - z + w}{-c_p - z + w} \right) F_{\Lambda^c} = \dim V_\lambda / l! \times \\ &F_{\Lambda^c} \bar{F}_M \left(\prod_{p=1, \dots, l} \frac{\sigma_{l+1,p} + \dots + \sigma_{l+m,p} + \sigma_{1p} + \dots + \sigma_{p-1,p} - 2c_p - z + w}{-c_p - z + w} \right) F_{\Lambda^c}; \end{aligned}$$

we have also used Proposition 1 of the present article, cf. [O]. Here in the last line the factors corresponding to $p = 1, \dots, l$ pairwise commute. By the same proposition applied to the partition γ instead of λ , any Young vector in the $\mathbb{C}S_{l+m}$ -module V_γ is an eigenvector for the action of the elements

$$\sigma_{l+1,p} + \dots + \sigma_{l+m,p} + \sigma_{1p} + \dots + \sigma_{p-1,p}; \quad p = 1, \dots, l.$$

The vector $Q \cdot v_\Gamma \in V_\gamma$ is a linear combination of the Young vectors, corresponding to standard tableaux obtained from Γ by permutations from the subgroup $S_\lambda \subset S_l$. The last expression for $F_{\Lambda^c} \bar{F}_M R_{\Lambda^c M}(z, w)$ now shows, in particular, that the number $r_\gamma(z, w)$ factorizes with respect to the columns of the Young diagram λ .

Firstly suppose that λ consists of one column only. Then the number $r_\gamma(z, w)$ is easy to evaluate; cf. [NT]. Here we have $c_i = 1 - i$ for each $i = 1, \dots, l$. Using the chain of subgroups $S_1 \subset S_2 \subset \dots \subset S_l$ corresponding to ordering $l, \dots, 1$ we then get

$$F_{\Lambda^c} \cdot \prod^{\rightarrow} (\sigma_{l+1,i} + \dots + \sigma_{l+m,i} - 1 + i + u) =$$

$$F_{\Lambda^c} \cdot \prod_{i=1, \dots, l} (\sigma_{l+1,i} + \dots + \sigma_{l+m,i} + \sigma_{li} + \dots + \sigma_{i+1,i} - 1 + l + u).$$

Here in the last line the factors corresponding to $i = 1, \dots, l$ pairwise commute. Their product commutes with any element from the subalgebra $\mathbb{C}S_l \subset \mathbb{C}S_{l+m}$, and acts on the vector $v_\Gamma \in V_\gamma$ as multiplication by the number

$$\prod_{i=1}^l (\mu_{a_i} - a_i + l + u). \quad (7)$$

Let us now apply this result to the j -th column of a general Young diagram λ , consecutively for $j = 1, \dots, \lambda_1$. For the general λ the content of the box (i, j) is $j - i$. According to our last expression for $F_{\Lambda^c} \bar{F}_M R_{\Lambda^c M}(z, w)$ in the product (7) we then have to replace l, μ_{a_i}, u by $\lambda'_j, \mu_{a_i} + j - 1, 2 - 2j - z + w$ respectively. Hence

$$r_\gamma(z, w) = \prod_{j=1}^{\lambda_1} \prod_{i=1}^{\lambda'_j} \frac{\mu_{a_i} + j - 1 - a_i + \lambda'_j + 2 - 2j - z + w}{i - j - z + w} \quad \square$$

§5. Choose any sequence $b_1, \dots, b_{\lambda_1} \in \{1, 2, \dots\}$ of pairwise distinct indices. Again, this sequence needs not to be increasing. Let us now regard the partition μ' conjugate to μ as an infinite sequence with finitely many parts. Determine an infinite sequence $\delta' = (\delta'_1, \delta'_2, \dots)$ by

$$\begin{aligned} \delta'_{b_j} &= \mu'_{b_j} + \lambda'_j; \quad j = 1, \dots, \lambda_1; \\ \delta'_b &= \mu'_b; \quad b \neq b_1, \dots, b_{\lambda_1}. \end{aligned}$$

Suppose $\delta'_1 \geq \delta'_2 \geq \dots$, so that δ' is a partition of $l + m$. Define δ as the partition conjugate to δ' . The irreducible $\mathbb{C}S_{l+m}$ -module V_δ appears in W with multiplicity one. Take the corresponding eigenvalue $r_\delta(z, w)$ of the operator $J : W \rightarrow W$.

Theorem 2:

$$r_\delta(z, w) = \prod_{(i,j)} \frac{z - w + \lambda_i + \mu'_{b_j} - i - b_j + 1}{z - w - i + j}$$

where the product is taken over all boxes (i, j) of the Young diagram λ .

Proof. Denote by Z_δ the minimal central idempotent in $\mathbb{C}S_{l+m}$ corresponding to the partition δ . Take the automorphism $*$ of the algebra $\mathbb{C}S_{l+m}$ such that $\sigma^* = \text{sgn}(\sigma)\sigma$; we have $Z_\delta^* = Z_\delta$, then. Reflecting the tableaux Λ and M in their main diagonals we get certain standard tableaux of shapes λ' and μ' respectively; denote these tableaux by Λ' and M' . Then we have $F_\Lambda^* = F_{\Lambda'}$ and $F_M^* = F_{M'}$.

On the other hand, by the definition of the number $r_\delta(z, w)$ we have the equality

$$Z_\delta F_\Lambda \bar{F}_M R_{\Lambda M}(z, w) = r_\delta(z, w) Z_\delta F_\Lambda \bar{F}_M.$$

By applying the automorphism $*$ to this equality we get

$$Z_\delta F_\Lambda \bar{F}_M R_{\Lambda M}^*(z, w) = r_\delta(z, w) Z_\delta F_\Lambda \bar{F}_M$$

But $R_{\Lambda\mu}^*(z, w) = R_{\Lambda'\mu'}(-z, -w)$ by definition. Therefore by applying Theorem 1 to the partitions λ', μ' instead of λ, μ and choosing $\gamma = \delta'$ we get

$$r_\delta(z, w) = \prod_{(i,j)} \frac{z - w + \lambda_j + \mu'_{b_i} - b_i - j + 1}{z - w + i - j}$$

where the product is taken over all boxes (i, j) of the diagram λ' . Equivalently, this product may be taken over all boxes (j, i) of the diagram λ \square

§6. Let us now derive the Corollary stated in the beginning of this article. We will use Theorems 1 and 2 in the simplest situation when $a_i = i$ for every $i = 1, \dots, \lambda'_1$ and $b_j = j$ for every $j = 1, \dots, \lambda_1$. Then we have $\gamma = \lambda + \mu$ and $\delta = (\lambda' + \mu')'$. By Theorems 1 and 2, $h_{\lambda\mu}(z, w) = r_{\lambda+\mu}(z, w) / r_{(\lambda'+\mu')'}(z, w)$ equals the product of the fractions

$$\frac{z - w - \lambda'_j - \mu_i + i + j - 1}{z - w + \lambda_i + \mu'_j - i - j + 1} \quad (8)$$

taken over all boxes (i, j) of the diagram λ . Consider those boxes of λ which do not belong to μ . These boxes form a skew Young diagram, let us denote it by ω . To obtain the Corollary, it suffices to prove the following

Proposition 7: *the product of the fractions (8) over the boxes (i, j) of ω equals 1.*

Proof. We will proceed by induction on the number of boxes in the diagram ω . Let us write u instead of $z - w$ for short. When the diagram ω is empty, the statement to prove is tautological. Now let (a, b) be any box of ω such that by removing it from λ we obtain again a Young diagram; then we have $\lambda_a = b$ and $\lambda'_b = a$. By applying the induction hypothesis to the last diagram instead of λ , we have to show that the product

$$\begin{aligned} & \prod_{j=\mu_a+1}^{b-1} \frac{u+b-1+\mu'_j-a-j+1}{u+b+\mu'_j-a-j+1} \cdot \prod_{i=\mu'_b+1}^{a-1} \frac{u-\mu_i-a+i+b-1}{u-\mu_i-a+1+i+b-1} \times \\ & \times \frac{u-\mu_a-a+a+b-1}{u+b+\mu'_b-a-b+1} \end{aligned} \quad (9)$$

equals 1. Note that here we have $\mu_a < \lambda_a$ and $\mu'_b < \lambda'_b$.

Suppose there is a box (\bar{i}, \bar{j}) of ω with $\mu_a < \bar{j} < \lambda_a$ and $\mu'_b < \bar{i} < \lambda'_b$, such that by adding this box to μ we obtain again a Young diagram. Then we have $\mu_{\bar{i}} = \bar{j} - 1$ and $\mu'_{\bar{j}} = \bar{i} - 1$. For the last diagram instead of μ , the product (9) equals 1 by the induction hypothesis. Then it suffices to check the equality to 1 of

$$\begin{aligned} & \frac{u+b-1+\bar{i}-1-a-\bar{j}+1}{u+b+\bar{i}-1-a-\bar{j}+1} \cdot \frac{u+b+\bar{i}-a-\bar{j}+1}{u+b+\bar{i}-a-\bar{j}} \times \\ & \frac{u-\bar{j}+1-a+\bar{i}+b-1}{u-\bar{j}+1-a+1+\bar{i}+b-1} \cdot \frac{u-\bar{j}-a+\bar{i}+b}{u-\bar{j}-a+\bar{i}+b-1} \end{aligned}$$

But this product has form $\frac{v-1}{v} \cdot \frac{v+1}{v} \cdot \frac{v}{v} \cdot \frac{v}{v}$ with $v = u - a + b + \bar{i} - \bar{j}$.

It remains to consider the case when there is no box (\bar{i}, \bar{j}) in ω with the above listed properties. Then $\mu'_j = a - 1$ for all $j = \mu_a + 1, \dots, b - 1$ and $\mu_i = b - 1$ for all $i = \mu'_b + 1, \dots, a - 1$. Hence in this remaining case the product (9) equals

$$\begin{aligned} & \frac{u + b - 1 + a - 1 - a - b + 1 + 1}{u + b + a - 1 - a - \mu_a - 1 + 1} \cdot \frac{u - b + 1 - a + \mu'_b + 1 + b - 1}{u - b + 1 - a + 1 + a - 1 + b - 1} \times \\ & \times \frac{u - \mu_a - a + a + b - 1}{u + b + \mu'_b - a - b + 1} = 1 \quad \square \end{aligned}$$

§7. In this final section we derive from Theorem 2 the formula $\dim V_\lambda = l! / h_\lambda$ for the dimension of the irreducible $\mathbb{C}S_l$ -module V_λ . We will actually show that the coefficient $l! / \dim V_\lambda$ in the relation $F_\Lambda^2 = l! / \dim V_\lambda \cdot F_\Lambda$ equals h_λ , the product of the l hook-lengths of the Young diagram λ . We will use induction on the number of rows in λ . If there is only one row in λ , then $h_\lambda = l!$ and $\dim V_\lambda = 1$, so the desired equality is clear. Let us now make the inductive assumption for λ , and consider the Young diagram obtained by adding m boxes to λ in the row $\lambda'_1 + 1$. Denote the new diagram by θ , we assume that $m \leq \lambda_i$ for any $i = 1, \dots, \lambda'_1$. Put $\mu = (m, 0, 0, \dots)$ and consider the eigenvalue $r_\theta(z, w)$ of the operator $J : W \rightarrow W$ corresponding to the multiplicity-free component $V_\theta \subset W$.

In our case, there is only one standard tableau M of shape μ . Let Θ be the unique standard tableau of shape θ agreeing with Λ in the entries $1, \dots, l$; the numbers $l + 1, \dots, l + m$ then appear in the last row of Θ . By definition,

$$F_\Theta \cdot F_\Lambda \bar{F}_M R_{\Lambda M}(z, w) = r_\theta(z, w) \cdot F_\Theta F_\Lambda \bar{F}_M = h_\lambda m! r_\theta(z, w) \cdot F_\Theta;$$

the second equality here has been obtained using the inductive assumption. On the other hand, due to Proposition 2 the matrix element F_Θ coincides with the value of the product $F_\Lambda \bar{F}_M R_{\Lambda M}(z, w)$ at $z - w = \lambda'_1$. To make the inductive step, it now remains to check that h_θ coincides with the value of $h_\lambda m! r_\theta(z, w)$ at $z - w = \lambda'_1$.

Let us use Theorem 2 when $b_j = j$ for every $j = 1, \dots, \lambda_1$. With our particular choice of μ , we then obtain that $r_\theta(z, w)$ equals the product over $i = 1, \dots, \lambda'_1$ of

$$\prod_{j=1}^m (z - w + \lambda_i - i - j + 2) \cdot \prod_{j=m+1}^{\lambda_i} (z - w + \lambda_i - i - j + 1) \cdot \prod_{j=1}^{\lambda_i} \frac{1}{z - w - i + j}.$$

Changing the running index j to $\lambda_i - j + 1$ in the products over $j = 1, \dots, m$ and over $j = m + 1, \dots, \lambda_i$ above, we obtain after cancellations the equality

$$r_\theta(z, w) = \prod_{i=1}^{\lambda'_1} \frac{z - w + \lambda_i - i + 1}{z - w + \lambda_i - m - i + 1}.$$

This equality shows that the value $r_\theta(z, w)$ at $z - w = \lambda'_1$ coincides with the ratio

$$\frac{h_\theta}{h_\lambda m!} = \prod_{i=1}^{\lambda'_1} \prod_{j=1}^m \frac{\lambda_i + \lambda'_1 - i - j + 2}{\lambda_i + \lambda'_1 - i - j + 1} = \prod_{i=1}^{\lambda'_1} \frac{\lambda'_1 + \lambda_i - i + 1}{\lambda'_1 + \lambda_i - m - i + 1} \quad \square$$

Let us make a few concluding remarks. Throughout §§1-6 we assumed that $z - w \notin \mathbb{Z}$, then the H_{l+m} -module W is irreducible. For $z - w \in \mathbb{Z}$, our Corollary implies that the module W is reducible if $z - w$ is a mixed hook-length of the second kind relative to λ and μ , and if $w - z$ is a mixed hook-length of the first kind. When $\lambda = \mu$, our Corollary implies that the module W is reducible if $|z - w|$ is a hook-length of λ . Moreover, then the module W is irreducible [LNT] for all remaining values $|z - w|$.

When $\lambda \neq \mu$ the H_{l+m} -module W maybe reducible while neither $z - w$ is a mixed hook-length of the second kind, nor $w - z$ is a mixed hook-length of the first kind. The irreducibility criterion for the module W with arbitrary λ and μ has been also given in [LNT]. This work shows that the module W is reducible if and only if the difference $z - w$ belongs to a certain finite subset $\mathcal{S}_{\lambda\mu} \subset \mathbb{Z}$ determined in [LZ]. This subset satisfies the property $\mathcal{S}_{\lambda\mu} = -\mathcal{S}_{\mu\lambda}$.

Denote by $\mathcal{D}_{\lambda\mu}$ the union of the sets of all zeroes and poles of the rational functions $r_{\lambda+\mu}(z, w)/r_\nu(z, w)$ in $z - w$, where ν ranges over all partitions γ and δ described in §3 and §5 respectively. Then $\mathcal{D}_{\lambda\mu} \subset \mathcal{S}_{\lambda\mu}$. Then $-\mathcal{D}_{\mu\lambda} \subset \mathcal{S}_{\lambda\mu}$ also. Using [LZ] one can demonstrate that if $\lambda'_1, \mu'_1 \leq 3$ then $\mathcal{D}_{\lambda\mu} \cup (-\mathcal{D}_{\mu\lambda}) = \mathcal{S}_{\lambda\mu}$. However, $\mathcal{D}_{\lambda\mu} \cup (-\mathcal{D}_{\mu\lambda}) \neq \mathcal{S}_{\lambda\mu}$ for general partitions λ and μ . For example, if $\lambda = (8, 3, 2, 1, 0, 0, \dots)$ and $\mu = (6, 4, 4, 0, 0, \dots)$ then $0 \in \mathcal{S}_{\lambda\mu}$ but $0 \notin \mathcal{D}_{\lambda\mu}, \mathcal{D}_{\mu\lambda}$.

For general λ and μ , it would be interesting to point out for every $t \in \mathcal{S}_{\lambda\mu}$ a partition ν of $l + m$, such that the $\mathbb{C}S_{l+m}$ -module V_ν appears in W with multiplicity one, and such that the ratio $r_{\lambda+\mu}(z, w)/r_\nu(z, w)$ has a zero or pole at t , as a rational function of $z - w$.

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